Last Time: $L: V \to W$ linear

$\ker(L) = \{v \in V : L(v) = 0_W\}$.

$\operatorname{ran}(L) = \{L(v) : v \in W\}$.

Prop: $L: V \to W$ linear.

① L is injective iff $\ker(L) = 0$

② L is surjective iff $\operatorname{ran}(L) = W$.

NB: A **bijective** linear map (i.e. a linear map which is both injective and surjective) is a linear **isomorphism**... Very important...

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Prop (Rank-Nullity Formula): Suppose $L: V \to W$ is a linear map. Then we have

$$\dim(V) = \underline{\dim(\ker(L))} + \dim(\operatorname{ran}(L)).$$

$\uparrow$ nullity(L) $\uparrow$ rank(L)

Pf: Let $L: V \to W$ be a linear map. Let B_0 be a basis for $\ker(L) \leq V$. Now B_0 extends to a basis $B \supseteq B_0$ for V. Let $A := B \backslash B_0$.

claim: $L(A) := \{L(a) : a \in A\} \subseteq \operatorname{ran}(L)$ is a basis of $\operatorname{ran}(L)$.

Note $L(A)$ spans $\operatorname{ran}(L)$ (because every element of $\operatorname{ran}(L)$ can be expressed as:

$$\boxed{L\left(\sum_{b \in B} c_b b\right)} = L\left(\sum_{b \in B_0} c_b b + \sum_{a \in A} c_a a\right)$$

$$= L\left(\sum_{b \in B_0} c_b b\right) + L\left(\sum_{a \in A} c_a a\right)$$

notation trick...
Point: Break up

$= \sum_{b \in B_0} c_b L(b) + \sum_{a \in A} c_a L(a)$

$$= 0_W + \sum_{a \in A} c_a L(a)$$

$$= \sum_{a \in A} c_a L(a)$$

b/c every elt of V can be expressed in this way (i.e. using basis B)

So $L(A)$ spans $\text{ran}(L)$. To see $L(A)$

is linearly indep., Suppose $\boxed{\sum_{i=1}^{n} c_i L(a_i) = 0_W.}$

Thus $L\left(\sum_{i=1}^{n} c_i a_i\right) = 0_W$, So $\sum_{i=1}^{n} c_i a_i \in \ker(L)$.

Hence $\sum_{i=1}^{n} c_i a_i + \boxed{\sum_{b \in B_0} 0 \cdot b}$ is the unique

expression for $\sum_{i=1}^{n} c_i a_i$ in terms of the basis B.

But $\sum_{i=1}^{n} c_i a_i \in \ker(L)$, So $c_i = 0$ for all i.

Hence $L(A)$ is linearly independent. Thus

$L(A)$ is a basis for $\text{ran}(L)$. But

$$B_0 \cup A = B, \quad \text{So} \quad \# B = \# B_0 + \# A.$$

On the other hand, $\# B = \dim(V)$, $\# B_0 = \dim(\ker(L))$

$\# L(A) = \dim(\text{ran}(L))$. Hence, we have

$$\boxed{\dim(V) = \dim(\ker(L)) + \# A} \quad \text{Now we must show}$$

$\# A = \# L(A)$. If $\# A > \# L(A)$, then there

are $a, a' \in A$ with $L(a) = L(a')$; But then $L(a - a') = 0_W$

So $a - a' \in \ker(L)$, so $B_0 \cup \{a, a'\}$ is linearly dependent, contradicting our assumption $B = B_0 \cup A \supseteq B_0 \cup \{a, a'\}$ is a basis... Thus $\# A \leq \# L(A) \leq \# A$.

$\hookrightarrow$ Hence
$$\dim(V) = \dim(\ker(L)) + \# A$$
$$= \dim(\ker(L)) + \# L(A)$$
$$= \dim(\ker(L)) + \dim(\operatorname{ran}(L))$$
$$= \text{nullity}(L) + \text{rank}(L).$$

Ex: Suppose $L : V \to \mathbb{R}^{15}$ has nullity$(L) = 7$ and L is surjective. Q: what is $\dim(V)$?

Sol: by the rank-nullity formula, $\dim(V) = \text{nullity}(L) + \text{rank}(L)$. nullity$(L) = 7$, and $\operatorname{ran}(L) = \mathbb{R}^{15}$, so rank$(L) = 15$.

Hence $\dim(V) = 7 + 15 = 22$.

Ex: Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is linear.
Q: what can rank(L) and nullity(L) be?

Sol: The rank-nullity formula yields
$$\left[3 = \dim(\mathbb{R}^3) = \text{nullity}(L) + \text{rank}(L) \right]$$
OTOH, rank$(L) \in \{0, 1, 2\}$. $\rightsquigarrow$ $\quad 0 \leq \text{rank}(L) \leq 2$

If rank$(L) = 1$: nullity$(L) = 3 - 1 = 2$
If rank$(L) = 2$: nullity$(L) = 3 - 2 = 1$
If rank$(L) = 0$: nullity$(L) = 3 - 0 = 3$

Thus $\boxed{1 \leq \text{nullity}(L) \leq 3}$.

Point: Every linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ has a nontrivial kernel!

In fact...

Cor: If ⟨$m < n$⟩ and $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then
L is not injective.

Pf: $\dim(\mathrm{dom}(L)) = \dim(\ker(L)) + \dim(\mathrm{ran}(L))$, so
$$n = \dim(\ker(L)) + \dim(\mathrm{ran}(L)).$$ Moreover,
$$0 \leq \dim(\mathrm{ran}(L)) \leq \dim(\mathbb{R}^m) = m \quad (\text{b/c } \mathrm{ran}(L) \leq \mathbb{R}^m).$$
Hence $n = \dim(\ker(L)) + \dim(\mathrm{ran}(L)) \leq \dim(\ker(L)) + m$.
So $0 < n - m \leq \dim(\ker(L))$. Hence $\ker(L) \neq \{0_v\}$,
so L is not injective. $\qquad\boxed{\mathrm{Z}}$

Ex: Let $L: V \to W$ be a linear map. Define for all
$U \leq W$, $L^{-1}U := \{v \in V : L(v) \in U\}$. Prove
$L^{-1}U \leq V$. **Q:** What can you say about $\dim(L^{-1}U)$?
  Hint: Rank nullity formula, apply to $L: L^{-1}U \to U$...

Lem: Suppose $L: V \to W$ and $Q: W \to U$ are linear.
Then $Q \circ L: V \to U$ is linear.
(i.e. compositions of linear maps are linear maps).

Recall: The <u>composition</u> of two functions $f: A \to B$
and $g: B \to C$ is the map $g \circ f: A \to C$
defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.
Remark: Composition of functions is associative...
  i.e. $h \circ (g \circ f) = (h \circ g) \circ f$.

Pf (Lem): Exercise. :) $\qquad\boxed{\mathrm{Z}}$

Point: Compositions of linear maps can be used to produce
   more linear maps :)

Def^n: A linear isomorphism of vector spaces V and W is a linear map $L: V \to W$ which is bijective. V and W are isomorphic when there is an isomorphism between them (and we write $\underline{V \cong W}$).

Ex: Claim $\mathbb{R}^4 \cong Mat_{2\times2}(\mathbb{R})$.

pf: We construct an explicit isomorphism.

Look at bases $\mathcal{E}_4 = \{e_1, e_2, e_3, e_4\}$ and

$$B = \left\{ b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, b_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Left to you: B is a basis of $Mat_{2\times2}(\mathbb{R})$.

[Define $L: \mathbb{R}^4 \to Mat_{2\times2}(\mathbb{R})$ by linearly extending $L(e_i) = b_i$ for $1 \leq i \leq 4$.] Left to you:

$$L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$ To see L is injective:

$$* \quad L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^* \iff \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff x = y = z = w = 0$$

$$\iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad \text{Hence } \ker(L) = 0.$$

To see L is surjective, note $ran(L) \supseteq B$, which is a basis for $Mat_{2\times2}(\mathbb{R})$, so $ran(L) = Mat_{2\times2}(\mathbb{R})$ yields L is surjective.

Hence L is bijective and linear, so L is an isomorphism, yielding $\mathbb{R}^4 \cong Mat_{2\times2}(\mathbb{R})$. ▨

NB: Nothing special about this example...
    All we needed to make this argument was
    that the vector spaces had the same dimension!

Prop: Two vector spaces are isomorphic if and
  only if they have the same dimension.
pf: Let V and W be vector spaces.
($\Longrightarrow$): Assume V and W are isomorphic. Thus there
is an isomorphism $L: V \to W$. Let B be a basis
of V. $L(B)$ is a basis for W by the
same argument we made when proving the
rank-nullity formula: $B = \emptyset \cup B$ and
$\emptyset$ is a basis for $\{0_v\} = \ker(L)$. Hence,
by injectivity $\dim(V) = \#B = \#L(B) = \dim(W)$.

($\Longleftarrow$): Assume V and W have the same dimension.
Let B be a basis of V and A a basis of W.
By assumption, $\#B = \dim(V) = \dim(W) = \#A$. Let
f be any bijection $f: B \to A$. Extend
f linearly to $F: V \to W$ (by a previous
proposition). Because A is a basis (hence
linearly independent), one can show $\ker(F) = 0$
(i.e., F is injective). OTOH $\operatorname{ran}(F) \supseteq F(B) = A$
so $\operatorname{ran}(F) = W$. Hence F is bijective. $\boxed{}$